MATH 207(c)–Analysis IV

1. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $T > 0$ such that, on the interval $[0, T)$, it takes the form

$$f(x) = \begin{cases} +1, & \frac{T}{2} \leqslant x < T, \\ -1, & 0 \leqslant x < \frac{T}{2}. \end{cases}$$

Compute the Fourier series of f.

2. Let us consider an integral equation of the following form:

$$u(t) = g(t) + \int_0^t k(t-s)u(s)\,ds.$$

In the above, $k, g : [0, +\infty) \to \mathbb{R}$ are given piecewise continuous functions and we are solving for a function $u : [0, +\infty) \to \mathbb{R}$. *Remark:* The above equation is a special case of a Volterra equation of second kind. These equations arise naturally in models dynamic systems where past values of a variable influence the current value with a weight determined by the *kernel* function k.

(a) Assuming that both g and k are such so that their Laplace transform is well-defined in some half-space of the form $\{z : \mathrm{Re}(z) > a\}$, find an expression for the Laplace transform of u.

(b) Find u in the case when $g(t) = t$ and $k(t) = e^{-t}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an *odd*, L-periodic function. Using Fourier series, find an odd and L-periodic solution u of the biharmonic equation

$$\frac{d^4u}{dx^4} = f.$$

4. Consider the following initial state on the interval $I = [0, 2L]$:

$$u_0(x) = \begin{cases} x, & 0 \leqslant x \leqslant L, \\ 2L - x, & L \leqslant x \leqslant 2L. \end{cases}$$

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0$$

and Dirichlet boundary conditions at $x = 0, 2L$:

$$u(0, t) = 0, \quad u(2L, t) = 0.$$

Hint: First extend u, u_0 as odd periodic functions in the variable $x \in \mathbb{R}$; what should be the the period for this extension?

5. For $\kappa > 0$, let us consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t), \quad t > 0\, x \in \mathbb{R}. \tag{1}$$

(a) Show that, for any solution u with $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm + \infty$ and any $t_2 \geqslant t_1$, we have

$$\int_{-\infty}^{+\infty} u(x, t_1)\, dx = \int_{-\infty}^{+\infty} u(x, t_2)\, dx.$$

(Hint: Compute the derivative $\partial_t \int_{-\infty}^{+\infty} u(x,t)\, dx$*.)*

(b) Compute the solution of (1) with initial data

$$u(x, 0) = \frac{1}{\sqrt{4\pi\tau\kappa}} e^{-\frac{x^2}{4\tau\kappa}}$$

for some given $\tau > 0$. Deduce, in particular, that the heat evolution of a Gaussian function is a Gaussian function at any fixed time. *(Hint: You will need to recall what is the Fourier transform of a Gaussian function, see Ex. 8.3)*

6. So far, we have only considered cases of *homogeneous* boundary conditions (namely boundary conditions which are invariant if we replace the unknown function $u(x,t)$ with $\lambda \cdot u(x,t)$; for example, Dirichlet conditions $u(x_0, t) = 0$ or Neumann conditions $\partial_x u(x_0, t) = 0$). Let us now consider the question of how to handle inhomogeneous boundary conditions.

To this end, let us consider the following inhomogeneous initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & \text{for } x \in (0,1),\, t > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), & \text{for } t > 0, \end{cases}$$

where $f : (0,1) \times (0, +\infty) \to \mathbb{R}$, $u_0 : (0,1) \to \mathbb{R}$ and $g_0, g_1 : [0, +\infty) \to \mathbb{R}$ are continuous functions.

Defining

$$a(x,t) = g_0(t) \cdot (1 - x) + g_1(t) \cdot x,$$

show that, if

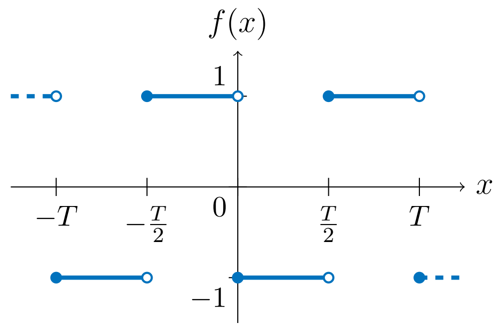
$$w(x,t) \doteq u(x,t) - a(x,t),$$

then w solves a heat equation with source term $f(x,t) - \frac{\partial a}{\partial t}(x,t) + \frac{\partial^2 a}{\partial x^2}(x,t)$ and *homogeneous* (in fact, Dirichlet) boundary conditions at $x = 0, 1$.

Solutions

1. We want to write the function $f(x)$ as a linear combination of the basis vectors $\cos\left(\frac{2\pi n}{T}x\right)$ and $\sin\left(\frac{2\pi n}{T}x\right)$ such that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)$ where T is the period of the function. We need to determine the coefficients a_n and b_n of this series. As we saw in the course, we extract these coefficients by projecting the function over the basis elements:

- $a_0 = \dfrac{2}{T}\langle\, f(x)\,, 1\,\rangle = \dfrac{2}{T}\displaystyle\int_0^T f(x)\,dx$
- $a_n = \dfrac{2}{T}\left\langle\, f(x)\,, \cos\left(\dfrac{2\pi n}{T}x\right)\right\rangle = \dfrac{2}{T}\displaystyle\int_0^L f(x)\cos\left(\dfrac{2\pi n}{T}x\right)dx$
- $b_n = \dfrac{2}{T}\left\langle\, f(x)\,, \sin\left(\dfrac{2\pi n}{T}x\right)\right\rangle = \dfrac{2}{T}\displaystyle\int_0^L f(x)\sin\left(\dfrac{2\pi n}{T}x\right)dx$



The function is odd. Therefore $a_n = 0\ \forall\, n \in \mathbb{N}$; we are only left to compute the sine coefficients :

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)dx = -\frac{2}{T}\int_0^{T/2}\sin\left(\frac{2\pi n}{T}x\right)dx + \frac{2}{T}\int_{T/2}^{T}\sin\left(\frac{2\pi n}{T}x\right)dx$$

$$\overset{*}{=} -\frac{2}{T}\int_0^{T/2}\sin\left(\frac{2\pi n}{T}x\right)dx + \frac{2}{T}\int_{-T/2}^{0}\sin\left(\frac{2\pi n}{T}x\right)dx \overset{**}{=} -\frac{4}{T}\int_0^{T/2}\sin\left(\frac{2\pi n}{T}x\right)dx$$

$$= \frac{2}{\pi n}\cos\left(\frac{2\pi n}{T}x\right)\Big|_0^{T/2} = \frac{2}{\pi n}(\cos(\pi n) - 1) = \frac{2}{\pi n}((-1)^n - 1) \overset{***}{=} -\frac{4}{\pi(2n+1)}$$

In $(*)$ we use the fact that our function is T-periodic to shift the interval of integration by an integer multiple of T This is eventually useful in $(**)$ where we merge the two integrals by using the fact that the function is odd. This decreases the number of integrals to compute. The final answer in $(***)$ is obtained by realizing first that $\cos(\pi n) = (-1)^n$, and second that the whole expression is null for even values of n. We only pick the odd values by replacing $n \to 2n+1$. Therefore, the final answer reads :

$$f(x) = -\sum_{n=1}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{2\pi(2n+1)}{T}x\right)$$

2. **(a)** We apply the Laplace transform on both sides of the equation. The integral term is clearly a convolution between $k(t)$ and $u(t)$.

$$\begin{aligned}\mathcal{L}[u(t)](z) &= \mathcal{L}[g(t)](z) + \mathcal{L}[k(t) * u(t)](z)\\ &= \mathcal{L}[g(t)](z) + \mathcal{L}[k(t)](z) \cdot \mathcal{L}[u(t)](z)\end{aligned}$$

For clarity, we write the Laplace transforms in upper cases and isolate $U(z)$:

$$U(z) = G(z) + K(z) \cdot U(z) \iff U(z) = \frac{G(z)}{1 - K(z)}$$

(b) In the case of $g(t) = t$ and $k(t) = e^{-t}$, we can compute the following form for $U(z)$:

$$U(z) = \frac{G(z)}{1-K(z)} = \frac{1}{z^2} \cdot \frac{1}{1 - \frac{1}{z+1}} = \frac{1}{z^2} + \frac{1}{z^3}$$

The inverse transform then reads :

$$\mathcal{L}^{-1}[U(z)](t) = \mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{z^3}\right](t) = t + \frac{t^2}{2} = u(t)$$

3. Since both f and the solution u that we are seeking are odd and L-periodic functions, their Fourier series consists only of the sine terms:

$$u(x) = \sum_{n=1}^{\infty} u_n \sin\left(\frac{2\pi n}{L}x\right), \quad f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{2\pi n}{L}x\right).$$

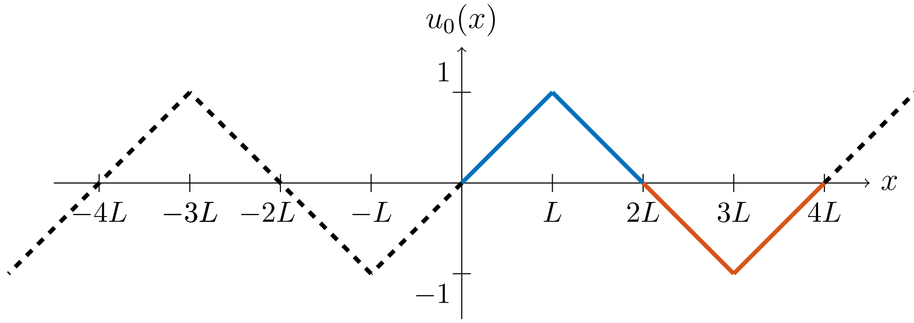
Injecting the above ansatz into the biharmonic equation, we have:

$$\sum_{n=1}^{\infty} u_n \left(\frac{2\pi n}{L}\right)^4 \sin\left(\frac{2\pi n}{L}x\right) = \equiv \sum_{n=1}^{\infty} f_n \sin\left(\frac{2\pi n}{L}x\right)$$

This equation is true only when the coefficients of the left and the right hand side match, hence the solution to this equation is a sine-Fourier series with coefficients u_n given in terms of the Fourier coefficients f_n of f by the formula:

$$u_n = f_n \left(\frac{L}{2\pi n}\right)^4.$$

4. As usual, since we want to solve a problem with Dirichlet boundary conditions, we extend our functions as **odd** functions of x with period twice the length of the initial x-interval; thus, the period in this case will be $T = 4L$. In the case of the initial profile $u_0(x)$, this extension will look as follows:



Being an odd, $4L$-periodic function of x, the function $u(x,t)$ decomposes as a sine-trigonometric series as follows:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{\pi n}{2L} x\right),$$

while the initial profile is expanded as:

$$u_0(x) = \sum_{n=1}^{\infty} b_{0n} \sin\left(\frac{\pi n}{2L} x\right),$$

where

$$\begin{aligned}
b_{0n} &= \frac{2}{4L} \int_0^{4L} u_0(x) \sin\left(\frac{\pi n}{2L} x\right) dx \\
&\overset{\text{u is odd}}{=} \frac{4}{4L} \int_0^{2L} u_0(x) \sin\left(\frac{\pi n}{2L} x\right) dx \\
&= \frac{1}{L} \left\{ \int_0^{L} x \sin\left(\frac{\pi n}{2L} x\right) dx + \int_L^{2L} (2L - x) \sin\left(\frac{\pi n}{2L} x\right) dx \right\} \\
&= \frac{1}{L} \left\{ \cdot \frac{-2L^2 \left[\pi n \cos\left(\frac{\pi n}{2}\right) - 2 \sin\left(\frac{\pi n}{2}\right)\right]}{\pi^2 n^2} + \frac{2L^2 (-1)^n \left[\pi n \cos\left(\frac{\pi n}{2}\right) - 2 \sin\left(\frac{\pi n}{2}\right)\right]}{\pi^2 n^2} \right\} \\
&= \frac{2L \left(\pi n \cos\left(\frac{\pi n}{2}\right) - 2 \sin\left(\frac{\pi n}{2}\right)\right)}{\pi^2 n^2} \cdot ((-1)^n - 1)
\end{aligned}$$

Note that the above expression can be simplified further, by considering the values of n modulo 4:

$$b_{0n} = \begin{cases} 0, & n = 4k \text{ or } n = 4k+2, \\ -\frac{4L}{\pi^2 n^2}, & n = 4k+1, \\ \frac{4L}{\pi^2 n^2}, & n = 4k+3. \end{cases}$$

Injecting the trigonometric series for u into the wave equation :

$$\frac{\partial^2}{\partial t^2}\left(\sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{\pi n}{2L}x\right)\right) = \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{\pi n}{2L}x\right)\right)$$

$$\iff \sum_{n=1}^{\infty} \underbrace{\left[b_n''(t) + b_n(t)\left(\frac{\pi n}{2L}\right)^2\right]}_{=0} \sin\left(\frac{\pi n}{2L}x\right) = 0$$

This expression must be valid for all x, and the only possibility is that the expression in square brackets is equal to zero. This gives us an ordinary differential equation on the coefficients $b_n(t)$:

$$b_n''(t) + \left(\frac{\pi n}{2L}\right)^2 b_n(t) = 0.$$

Since $t \in [0, +\infty)$, we can naturally linearize and solve this equation with the Laplace transform. We write the Laplace transform in upper case for clarity.

$$\mathcal{L}\left[b_n''(t) + b_n(t)\left(\frac{\pi n}{2L}\right)^2\right] = 0 \iff z^2 B_n(z) - z b_n(0) - b_n'(0) + \left(\frac{\pi n}{2L}\right)^2 \cdot B_n(z) = 0$$

The factors $b_n(0)$ and $b_n'(0)$ correspond to the Fourier coefficients that describe the function $u(x,t)$ and its derivative $\partial_t u(x,t)$ at $t = 0$, respectively. Since $u(x,0) = u_0(x)$ and $\frac{\partial u}{\partial t}(x,0)$, we must have

$$b_n(0) = b_{0n} \quad \text{and} \quad b_n'(0) = 0.$$

Thus, we have

$$z^2 B_n(z) - z b_{0n} + \left(\frac{\pi n}{2L}\right)^2 \cdot B_n(z) = 0 \iff B_n(z) = b_{0n}\frac{z}{z^2 + \left(\frac{\pi n}{2L}\right)^2}$$

Hence, since $\mathcal{L}\left[\cos\left(\frac{\pi n}{2L}t\right)\right](z) = \frac{z}{z^2 + \left(\frac{\pi n}{2L}\right)^2}$:

$$b_n(t) = b_{0n}\cos\left(\frac{\pi n}{2L}t\right).$$

The complete solution for $u(x,t)$ is therefore :

$$u(x,t) = \sum_{n=1}^{\infty} b_{0n}\cos\left(\frac{\pi n}{2L}t\right)\sin\left(\frac{\pi n}{2L}x\right).$$

5. **(a)** We compute the following derivative :

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} u(x,t)\,dx = \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t}(x,t)\,dx \overset{\text{Equation for } u}{=} \int_{-\infty}^{+\infty} \kappa \frac{\partial^2 u}{\partial x^2}(x,t)\,dx = \kappa\,\frac{\partial u}{\partial x}(x,t)\Big|_{-\infty}^{+\infty} = 0$$

Since $\partial_t \int_{-\infty}^{+\infty} u(x,t)\,dx = 0$, it follows that $\int_{-\infty}^{+\infty} u(x,t_1)\,dx = \int_{-\infty}^{+\infty} u(x,t_2)\,dx$ for all $t_1, t_2 \geqslant 0$.

(b) We apply the Fourier transform in x on both sides of the heat equation. We use the notation $\mathcal{F}[u(x,t)](a,t) = \hat{u}(a,t)$

$$\frac{\partial \hat{u}}{\partial t}(a,t) = \kappa(ia)^2\hat{u}(a,t) \iff \frac{\partial \hat{u}}{\partial t}(a,t) = -\kappa a^2\hat{u}(a,t)$$

We can solve the above ODE in time using, for instance, a Laplace transform in t; we obtain:

$$\hat{u}(a,t) = \hat{u}(a,0)e^{-\kappa a^2 t}.$$

Using the initial condition $u(x,0) = \frac{e^{-\frac{x^2}{4\tau\kappa}}}{\sqrt{4\pi\kappa\tau}}$, we compute

$$\hat{u}(a,0) = \mathcal{F}\left[\frac{e^{-x^2/4\tau\kappa}}{\sqrt{4\pi\tau\kappa}}\right] = \frac{e^{-a^2\kappa\tau}}{\sqrt{2\pi}}$$

(in the above, we used the fact that $\mathcal{F}[e^{-\frac{x^2}{2}}](a) = e^{-\frac{a^2}{2}}$ together with the rescaling property of the Fourier transform: $\mathcal{F}[f(\lambda x)](a) = \frac{1}{|\lambda|}\mathcal{F}[f(x)](\frac{a}{\lambda})$). Therefore,

$$\hat{u}(a,t) = \frac{e^{-\kappa a^2(t+\tau)}}{\sqrt{2\pi}}.$$

The original function $u(x,t)$ is then given by the inverse Fourier transform :

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(a,t)](x,t) = \mathcal{F}^{-1}\left[\frac{e^{-\kappa a^2(t+\tau)}}{\sqrt{2\pi}}\right] = \frac{e^{-x^2/4\kappa(t+\tau)}}{2\sqrt{\pi\kappa(t+\tau)}}$$

6. The new function $w(x,t) = u(x,t) - a(x,t)$ has indeed homogeneous (in fact Dirichlet) boundary conditions at $x = 0, 1$ since :

- $w(0,t) = u(0,t) - g_0(t)(1-0) - \cancelto{0}{g_1(t)\cdot 0} = g_0(t) - g_0(t) = 0$
- $w(1,t) = u(1,t) - \cancelto{0}{g_0(t)(1-1)} - g_1(t)\cdot 1 = g_1(t) - g_1(t) = 0$

Using the fact that u satisfies

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t),$$

we can then calculate:

$$\begin{aligned}\frac{\partial w}{\partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) &= \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial a}{\partial t}(x,t) + \frac{\partial^2 a}{\partial x^2}(x,t) \\ &= f(x,t) - \frac{\partial}{\partial t}(g_0(t)(1-x) + g_1(t)x) + \frac{\partial^2}{\partial x^2}(g_0(t)(1-x) + g_1(t)x) \\ &= f(x,t) - g_0'(t)(1-x) - g_1'(t)x.\end{aligned}$$

Overall, setting

$$F(x,t) \doteq f(x,t) - g_0'(t)(1-x) - g_1'(t)x \quad \text{and} \quad w_0(x) \doteq u_0(x) - g_0(0)(1-x) - g_1(0)x,$$

the function $w(x,t)$ solves the initial-boundary value problem with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) = F(x,t), \quad t > 0, x \in (0,1), \\ w(x,0) = w_0(x), \\ w(0,t) = 0 = w(1,t). \end{cases}$$

The above is of the same type as the ones we saw how to solve in the lectures.